

# Some new bounds for the Hadamard product and the Fan product of matrices<sup>★</sup>

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## Abstract

If  $A$  and  $B$  are nonnegative matrices, a sharp upper bound on the spectral radius  $\rho(A \circ B)$  for the Hadamard product of two nonnegative matrices is given, and the minimum eigenvalue  $\tau(A \star B)$  of the Fan product of two  $M$ -matrices  $A$  and  $B$  is discussed. In addition, we also give a sharp lower bound on  $\tau(A \circ B^{-1})$  for the Hadamard product of  $A$  and  $B^{-1}$ . Several examples, illustrating that the given bound is stronger than the existing bounds, are also given.

*Key words:* Hadamard product; Nonnegative matrices; Spectral radius; Fan product;  $M$ -matrix; Inverse  $M$ -matrix; Minimum eigenvalue

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## 1 Introduction

In this paper, for a positive integer  $n$ ,  $N$  denotes the set  $\{1, 2, \dots, n\}$ .  $\mathbb{R}^{n \times n}$  denotes the set of all  $n \times n$  real matrices and the set of all  $n \times n$  complex matrices is denoted by  $\mathbb{C}^{n \times n}$ . Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be two real  $n \times n$  matrices. We write  $A \geq B (> B)$  if  $a_{ij} \geq b_{ij} (> b_{ij})$  for all  $i, j \in N$ . If  $A \geq 0 (> 0)$ , we say that  $A$  is a nonnegative (positive) matrix. The spectral radius of  $A$  is denoted by  $\rho(A)$ . If  $A$  is a nonnegative matrix, the Perron-Frobenius theorem guarantees that  $\rho(A) \in \sigma(A)$ , where  $\sigma(A)$  is the set of all eigenvalues of  $A$ . In addition, define  $\tau(A) \triangleq \min\{|\lambda| \mid \lambda \in \sigma(A)\}$ , and denote by  $\mathcal{M}_n$  the set of nonsingular  $M$ -matrices (see [1]).

For  $n \geq 2$ , an  $n \times n$  matrix  $A$  is said to be reducible if there exists a permutation matrix  $P$  such that

$$P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

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where  $B$  and  $D$  are square matrices of order at least one. If no such permutation matrix exists, then  $A$  is called irreducible. If  $A$  is a  $1 \times 1$  complex matrix, then  $A$  is irreducible if and only if its single entry is nonzero (see [2]).

According to Ref. [2], a matrix  $A$  is called an  $M$ -matrix, if there exists an  $n \times n$  nonnegative real matrix  $P$  and a nonnegative real number  $\alpha$  such that  $A = \alpha I - P$ , and  $\alpha \geq \rho(P)$ , where  $\rho(P)$  denotes the spectral radius of  $P$  and  $I$  is the identity matrix. Moreover, if  $\alpha > \rho(P)$ ,  $A$  is called a nonsingular  $M$ -matrix; if  $\alpha = \rho(P)$ , we call  $A$  a singular  $M$ -matrix.

In addition, a matrix  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is called  $Z$ -matrix if all of whose off-diagonal entries are negative, and denoted by  $A \in \mathcal{Z}_n$ . For convenience, the following simple facts are needed (see Problems 16, 19 and 28 in Section 2.5 of [3]):

- (1)  $\tau(A) \in \sigma(A)$ ;
- (2) If  $A, B \in \mathcal{M}_n$ , and  $A \geq B$ , then  $\tau(A) \geq \tau(B)$ ;
- (3) If  $A \in \mathcal{M}_n$ , then  $\rho(A^{-1})$  is the Perron eigenvalue of the nonnegative matrix  $A^{-1}$ , and  $\tau(A) = \frac{1}{\rho(A^{-1})}$  is a positive real eigenvalue of  $A$ .

Let  $A$  be an irreducible nonsingular  $M$ -matrix. It is well known that there exist positive vectors  $u$  and  $v$  such that  $Au = \tau(A)u$  and  $v^T A = \tau(A)v^T$ , where  $u$  and  $v$  are right and left Perron eigenvectors of  $A$ , respectively.

The Hadamard product of  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$  and  $B = (b_{ij}) \in \mathbb{C}^{n \times n}$  is defined by  $A \circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{n \times n}$ .

For two real matrices  $A, B \in \mathcal{M}_n$ , the Fan product of  $A$  and  $B$  is denoted by  $A \star B = C = [c_{ij}] \in \mathcal{M}_n$  and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

We define: for any  $i, j, l \in N$ ,

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l, i} |a_{lk}|}, \quad l \neq i; \quad r_i = \max_{l \neq i} \{r_{li}\}, \quad i \in N,$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j, i} |a_{jk}|r_k}{|a_{jj}|}, \quad j \neq i; \quad s_i = \max_{j \neq i} \{s_{ji}\}, \quad i \in N,$$

throughout the paper.

For two nonnegative matrices  $A, B$ , we will exhibit a new upper bound for  $\rho(A \circ B)$ , a new lower bound on the eigenvalue  $\tau(A \star B)$  for the Fan product and a new lower bound on the eigenvalue  $\tau(A \circ B^{-1})$  for the hadamard product in this paper.

## 2 An upper bound for the spectral radius of the Hadamard product of two nonnegative matrices

In ([3], p. 358), there is a simple estimate for  $\rho(A \circ B)$ : if  $A, B \in \mathbb{R}^{n \times n}$ ,  $A \geq 0$ , and  $B \geq 0$ , then

$$\rho(A \circ B) \leq \rho(A)\rho(B). \quad (2.1)$$

Fang [9] gave an upper bound for  $\rho(A \circ B)$ , that is,

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \left\{ 2a_{ii}b_{ii} + \rho(A)\rho(B) - b_{ii}\rho(A) - a_{ii}\rho(B) \right\}, \quad (2.2)$$

which is shaper than the bound  $\rho(A)\rho(B)$  in ([3], p. 358).

Recently, Liu [1] improved the above results, have

$$\begin{aligned} \rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \right. \\ \left. + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})] \right\}^{\frac{1}{2}}. \end{aligned} \quad (2.3)$$

Firstly, we give some lemmas in this section.

**Lemma 2.1** (*Perron-Frobenius theorem*)([3]). *If  $A$  is an irreducible nonnegative matrix, there exist positive vectors  $u$ , such that  $Au = \rho(A)u$ .*

**Lemma 2.2** ([3]). *If  $A, B \in \mathbb{C}^{n \times n}$ ,  $D$  and  $E$  are positive diagonal matrices, then*

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

**Lemma 2.3** (*Brauer's theorem*). *Let  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$  ( $n \geq 2$ ), then all the eigenvalues of  $A$  lie inside the union of  $\frac{n(n-1)}{2}$  ovals of Cassini, i.e.,*

$$B(A) = \bigcup_{i,j=1; i \neq j}^n \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \left( \sum_{k \neq i} |a_{ki}| \right) \left( \sum_{k \neq j} |a_{kj}| \right) \right\}, \quad (2.4)$$

Obviously, if we denote  $C = D^{-1}AD$ ,  $D = \text{diag}(d_1, d_2, \dots, d_n)$ ,  $d_i > 0$ , then  $C$  and  $A$  have the same eigenvalues, we obtain that all the eigenvalues of  $A$  lie in the region:

$$\bigcup_{i,j=1; i \neq j}^n \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \left( \sum_{k \neq i} \frac{d_k}{d_i} |a_{ik}| \right) \left( \sum_{k \neq j} \frac{d_l}{d_j} |a_{jl}| \right) \right\}. \quad (2.5)$$

Next, we present a new estimating formula on the upper bound of  $\rho(A \circ B)$ .

**Theorem 2.1** *If  $A = (a_{ij})$  and  $B = (b_{ij})$  are nonnegative matrices,  $s_i = \max_{j \neq i} \{a_{ij}\}$ ,*

$t_i = \max_{j \neq i} \{b_{ij}\}$ , then

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\}. \quad (2.6)$$

**Proof.** It is evident that the inequality (2.6) holds with the equality for  $n = 1$ . Therefore, we assume that  $n \geq 2$  and divide two cases to prove this problem.

**Case 1.** Suppose that  $A \circ B$  is irreducible. Obviously  $A$  and  $B$  are also irreducible. By Lemma 2.1, there exists positive vectors  $u = (u_1, u_2, \dots, u_n)$  and have

$$(D^{-1}AD)u = \rho(D^{-1}AD)u = \rho(A)u,$$

where  $D = \text{diag}(d_1, d_2, \dots, d_n)$ ,  $d_i > 0$ , then

$$\sum_{j \neq i} \frac{a_{ij}d_j u_j}{d_i u_i} = \rho(A) - a_{ii}.$$

Define  $U = \text{diag}(u_1, u_2, \dots, u_n)$ ,  $C = (DU)^{-1}A(DU)$ , then we have that

$$C = \begin{pmatrix} a_{11} & \frac{d_2 u_2}{d_1 u_1} a_{12} & \cdots & \frac{d_n u_n}{d_1 u_1} a_{1n} \\ \frac{d_1 u_1}{d_2 u_2} a_{21} & a_{22} & \cdots & \frac{d_n u_n}{d_2 u_2} a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_1 u_1}{d_n u_n} a_{n1} & \frac{d_2 u_2}{d_n u_n} a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is an irreducible nonnegative matrix and

$$C \circ B = (m_{ij}) = \begin{pmatrix} a_{11}b_{11} & \frac{d_2 u_2}{d_1 u_1} a_{12}b_{12} & \cdots & \frac{d_n u_n}{d_1 u_1} a_{1n}b_{1n} \\ \frac{d_1 u_1}{d_2 u_2} a_{21}b_{21} & a_{22}b_{22} & \cdots & \frac{d_n u_n}{d_2 u_2} a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_1 u_1}{d_n u_n} a_{n1}b_{n1} & \frac{d_2 u_2}{d_n u_n} a_{n2}b_{n2} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

By Lemma 2.2,

$$(DU)^{-1}(A \circ B)(DU) = (DU)^{-1}A(DU) \circ B = C \circ B,$$

i.e.,  $\rho(A \circ B) = \rho(C \circ B)$ .

By the inequality (2.4) and  $\rho(A \circ B) \geq a_{ii}b_{ii}$  (see [5]), for any  $j \neq i \in N$ ,

we have

$$\begin{aligned}
(\rho(A \circ B) - a_{ii}b_{ii})(\rho(A \circ B) - a_{jj}b_{jj}) &\leq \sum_{k \neq i} |m_{ik}| \sum_{l \neq j} |m_{jl}| \\
&= \sum_{k \neq i} \frac{d_k u_k a_{ik} b_{ik}}{d_i u_i} \sum_{l \neq j} \frac{d_l u_l a_{jl} b_{jl}}{d_j u_j} \\
&\leq \left( \max_{k \neq i} \{b_{ik}\} \sum_{k \neq i} \frac{d_k u_k a_{ik}}{d_i u_i} \right) \left( \max_{l \neq j} \{a_{jl}\} \sum_{l \neq j} \frac{d_l u_l b_{jl}}{d_j u_j} \right) \\
&\leq \max_{k \neq i} \{b_{ik}\} (\rho(A) - a_{ii}) \max_{l \neq j} \{a_{jl}\} (\rho(B) - b_{jj}) \\
&= t_i s_j (\rho(A) - a_{ii}) (\rho(B) - b_{jj}).
\end{aligned} \tag{2.7}$$

Thus, by solving the quadratic inequality (2.7), we have that

$$\begin{aligned}
\rho(A \circ B) &\leq \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\} \\
&\leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\}.
\end{aligned}$$

i.e., the conclusion (2.6) holds.

**Case 2.** If  $A \circ B$  is reducible. We may denote by  $P = (p_{ij})$  the  $n \times n$  permutation matrix  $(p_{ij})$  with

$$p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1,$$

the remaining  $p_{ij}$  zero, then both  $A + \varepsilon P$  and  $B + \varepsilon P$  are nonnegative irreducible matrices for any sufficiently small positive real number  $\varepsilon$ . Now we substitute  $A + \varepsilon P$  and  $B + \varepsilon P$  for  $A$  and  $B$ , respectively in the previous Case 1, and then letting  $\varepsilon \rightarrow 0$ , the result (2.6) follows by continuity.  $\square$

**Remark 2.1.** Next, we give a comparison between the upper bound in the inequality (2.3) and the upper bound in the inequality (2.6). Without loss of generality, if  $t_i + b_{ii} \geq \rho(B)$ ,  $s_j + a_{jj} \geq \rho(A)$ ,  $i, j = 1, \dots, n$ , then we have  $t_i s_j \geq (\rho(B) - b_{ii})(\rho(A) - a_{jj})$ . Thus, the upper bound in the inequality (2.6) is better than the upper bound in the inequality (2.3).

**Example 2.1** . Let  $A$  and  $B$  be the same as in Example 1 from [1]:

$$A = (a_{ij}) = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 0.05 & 1 & 1 \\ 0 & 1 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{pmatrix}, \quad B = (b_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

By direct calculation,  $\rho(A \circ B) = 5.7339$ .

According to (2.1), we have

$$\rho(A \circ B) \leq \rho(A)\rho(B) = 22.9336.$$

If we apply (2.2) and (2.3), we get

$$\rho(A \circ B) \leq \max_{1 \leq i \leq 4} \left\{ 2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A) \right\} = 17.1017,$$

and

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{1/2} \right\} = 11.6478.$$

If we apply Theorem 2.1, we obtain that

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{1/2} \right\} = 8.1897.$$

The example shows that the bound in Theorem 2.1 is better than the existing bounds.

In addition, by the Theorem 2.1 and [1], we also have the following corollary:

**Corollary 2.1** *Let  $A$  and  $B$  be nonnegative matrices, then*

$$\begin{aligned} |\det(A \circ B)| &\leq (\rho(A \circ B))^n \\ &\leq \max_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{1/2} \right\}^n \\ &\leq \max_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{1/2} \right\}^n. \end{aligned}$$

### 3 Inequalities for the Fan product of two $M$ -matrices

It is known (p.359, [3]) that the following classical result is given: if  $A, B \in \mathbb{R}^{n \times n}$  are  $M$ -matrices, then

$$\tau(A \star B) \geq \tau(A)\tau(B). \quad (3.1)$$

In 2007, Fang improved (3.1) in the Remark 3 of Ref. [9] and gave a new lower bound for  $\tau(A \star B)$ , that is

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ b_{ii}\tau(A) + a_{ii}\tau(B) - \tau(A)\tau(B) \right\}. \quad (3.2)$$

Subsequently, Liu et al.[1] gave a sharper bound than (3.2), i.e.,

$$\begin{aligned} \tau(A \star B) &\geq \frac{1}{2} \min_{i \neq j} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(b_{ii} - \tau(B))(a_{ii} - \tau(A))(b_{jj} - \tau(B))(a_{jj} - \tau(A))]^{1/2} \right\}. \end{aligned} \quad (3.3)$$

In addition, by the definition of Fan product, the following lemma holds:

**Lemma 3.1** ([1]). If  $A, B \in \mathbb{C}^{n \times n}$  be nonsingular  $M$ -matrices,  $D$  and  $E$  are positive diagonal matrices, then

$$D(A \star B)E = (DAE) \star B = (DA) \star (BE) = (AE) \star (DB) = A \star (DBE).$$

Next, we give a new lower bound on the minimum eigenvalue  $\tau(A \star B)$  of the Fan product of nonsingular  $M$ -matrices.

**Theorem 3.1** If  $A = (a_{ij})$  and  $B = (b_{ij})$  are nonsingular  $M$ -matrices,  $s_i = \max_{j \neq i} |a_{ij}|$ ,  $t_i = \max_{j \neq i} |b_{ij}|$ , then

$$\begin{aligned} \tau(A \star B) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \Big\}. \end{aligned} \quad (3.4)$$

**Proof.** It is clear that the (3.4) holds with the equality for  $n = 1$ .

We next assume  $n \geq 2$  and divide two cases to prove this problem.

**Case 1.** Suppose that  $A \star B$  is irreducible. Obviously  $A$  and  $B$  are also irreducible. By [5], there exists positive vectors  $u = (u_1, u_2, \dots, u_n)$  such that

$$(D^{-1}AD)u = \tau(D^{-1}AD)u = \tau(A)u,$$

where  $D = \text{diag}(d_1, d_2, \dots, d_n)$ ,  $d_i > 0$ , and then

$$a_{ii} - \sum_{j \neq i} \frac{|a_{ij}|d_j u_j}{d_i u_i} = \tau(A).$$

Define  $U = \text{diag}(u_1, u_2, \dots, u_n)$ ,  $C = (DU)^{-1}A(DU)$ , we have that

$$C = \begin{pmatrix} a_{11} & \frac{d_2 u_2}{d_1 u_1} a_{12} & \cdots & \frac{d_n u_n}{d_1 u_1} a_{1n} \\ \frac{d_1 u_1}{d_2 u_2} a_{21} & a_{22} & \cdots & \frac{d_n u_n}{d_2 u_2} a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_1 u_1}{d_n u_n} a_{n1} & \frac{d_2 u_2}{d_n u_n} a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is an irreducible nonsingular  $M$ - matrix, then

$$C \star B = (m_{ij}) = \begin{pmatrix} a_{11}b_{11} & \frac{d_2 u_2}{d_1 u_1} a_{12}b_{12} & \cdots & \frac{d_n u_n}{d_1 u_1} a_{1n}b_{1n} \\ \frac{d_1 u_1}{d_2 u_2} a_{21}b_{21} & a_{22}b_{22} & \cdots & \frac{d_n u_n}{d_2 u_2} a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_1 u_1}{d_n u_n} a_{n1}b_{n1} & \frac{d_2 u_2}{d_n u_n} a_{n2}b_{n2} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

By the Lemma 3.1,

$$(DU)^{-1}(A \star B)(DU) = (DU)^{-1}A(DU) \star B = C \star B,$$

i.e.,  $\tau(A \star B) = \tau(C \star B)$ .

In addition, by the inequality (2.4) and  $0 \leq \tau(A \star B) \leq a_{ii}b_{ii}$  (see [5]), for any  $j \neq i \in N$ , we have

$$\begin{aligned}
|\tau(A \star B) - a_{ii}b_{ii}| |\tau(A \star B) - a_{jj}b_{jj}| &\leq \sum_{k \neq i} |m_{ik}| \sum_{l \neq j} |m_{jl}| \\
&= \sum_{k \neq i} \left| \frac{d_k u_k a_{ik} b_{ik}}{d_i u_i} \right| \sum_{l \neq j} \left| \frac{d_l u_l a_{jl} b_{jl}}{d_j u_j} \right| \\
&\leq \left( \max_{k \neq i} |b_{ik}| \sum_{k \neq i} \left| \frac{d_k u_k a_{ik}}{d_i u_i} \right| \right) \left( \max_{l \neq j} |a_{jl}| \sum_{l \neq j} \left| \frac{d_l u_l b_{jl}}{d_j u_j} \right| \right) \\
&\leq \max_{k \neq i} |b_{ik}| (a_{ii} - \tau(A)) \max_{l \neq j} |a_{jl}| (b_{jj} - \tau(B)) \\
&= t_i s_j (a_{ii} - \tau(A)) (b_{jj} - \tau(B)).
\end{aligned} \tag{3.5}$$

Thus, by solving the quadratic inequality (3.5), we have that

$$\begin{aligned}
\tau(A \star B) &\geq \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\} \\
&\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}.
\end{aligned}$$

i.e., the conclusion (3.4) holds.

**Case 2.** If  $A \star B$  is reducible. It is well known that a matrix in  $\mathcal{Z}_n$  is a nonsingular  $M$ -matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [5]). We denote by  $P = (p_{ij})$  the  $n \times n$  permutation matrix  $(p_{ij})$  with

$$p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1,$$

the remaining  $p_{ij}$  zero, then both  $A - \varepsilon P$  and  $B - \varepsilon P$  are irreducible nonsingular  $M$ -matrices for any sufficiently small positive real number  $\varepsilon$ . Now we substitute  $A - \varepsilon P$  and  $B - \varepsilon P$  for  $A$  and  $B$ , respectively in the previous Case 1, and then letting  $\varepsilon \rightarrow 0$ , the result (3.4) follows by continuity.  $\square$

**Remark 3.1.** Similarly, we give a comparison between the lower bound in the inequality (3.3) and the lower bound in the inequality (3.4). If  $a_{jj} \geq \tau(A) + s_j$ ,  $b_{ii} \geq \tau(B) + t_i$ ,  $i, j = 1, \dots, n$ , then  $(a_{jj} - \tau(A))(b_{ii} - \tau(B)) \geq s_j t_i$  for all  $i \neq j$ . Thus, the lower bound in the inequality (3.4) is better than the lower bound in the inequality (3.3).

In addition, from Theorem 3.1 and [5], we may get the following corollary.

**Corollary 3.1** . If  $A, B$  are nonsingular  $M$ -matrices, then

$$\begin{aligned}
|\det(A \star B)| &\geq \left( \tau(A \star B) \right)^n \\
&\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}^n \\
&\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(a_{ii} - \tau(A))(b_{ii} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}^n.
\end{aligned}$$

**Example 3.1** ([1]). Let  $A$  and  $B$  be the nonsingular  $M$ -matrices:



$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & -1 & 2 \end{pmatrix}, \quad B = (b_{ij}) = \begin{pmatrix} 1 & -0.25 & -0.25 \\ -0.5 & 1 & -0.25 \\ -0.25 & -0.5 & 1 \end{pmatrix}.$$

By (3.1), we have

$$\tau(A \star B) \geq \tau(A)\tau(B) = 0.1854.$$

If we use the inequalities (3.2) and (3.3), then we get

$$\tau(A \star B) \geq \min_{1 \leq i \leq 3} \{a_{ii}\tau(B) + b_{ii}\tau(A) - \tau(A)\tau(B)\} = 0.6980,$$

and

$$\begin{aligned} \tau(A \star B) \geq \min_{i \neq j} \frac{1}{2} \{ & a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4(a_{ii} - \tau(A))(b_{ii} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \} = 0.7655. \end{aligned}$$

If we apply Theorem 3.1, we obtain that

$$\begin{aligned} \tau(A \star B) \geq \min_{i \neq j} \frac{1}{2} \{ & a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \} = 0.8002. \end{aligned}$$

In fact,  $\tau(A \star B) = 0.8819$ . The example shows that the bound in Theorem 3.1 is better than the existing bounds.

#### 4 A bound for the Hadamard product of $M$ -matrix and an inverse $M$ -matrix

Now, we consider the lower bound of  $\tau(A \circ B^{-1})$ , for  $A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}_n$  and  $B^{-1} = (\beta_{ij})$ .

Firstly, in [3], Horn and Johnson gave the classical results

$$\tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii}. \quad (4.1)$$

Subsequently, Huang [8] gave new bound for  $\tau(A \circ B^{-1})$ , that is,

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}, \quad (4.2)$$

where  $\rho(J_A)$  and  $\rho(J_B)$  are the spectral radius of the Jacobi iterative matrices  $J_A$  and  $J_B$ , respectively.

In 2008, Li [10] improved the above results as follows.

$$\tau(A \circ B^{-1}) \geq \min_i \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}}. \quad (4.3)$$

Recently, Chen [11] improved the result and gave a new lower bound for  $\tau(A \circ B^{-1})$ :

$$\begin{aligned} \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 \\ & + 4a_{ii}a_{jj}\beta_{ii}\beta_{jj}\rho^2(J_A)\rho^2(J_B)]^{\frac{1}{2}} \Big\}. \end{aligned} \quad (4.4)$$

In this section, we give a lower bound of  $\tau(A \circ B^{-1})$  for  $M$ -matrix and inverse  $M$ -matrix, which improves the above bounds.

**Lemma 4.1** ([12]). *If  $A = (a_{ij}) \in \mathcal{M}_n$ , there exists a positive diagonal matrix  $D$  such that  $D^{-1}AD$  is a strictly row diagonally dominant matrix.*

**Lemma 4.2** ([12]). *If  $A = (a_{ij}) \in \mathcal{M}_n$ , and  $D = \text{diag}(d_1, d_2, \dots, d_n)$ ,  $d_i > 0$  ( $i \in N$ ), then  $D^{-1}AD$  is also an  $M$ -matrix.*

**Lemma 4.3** ([12]). *If  $A, B \in \mathcal{M}_n$ , then  $B \circ A^{-1}$  is also an  $M$ -matrix.*

**Lemma 4.4** ([10]). *If  $A = (a_{ij})$  be a strictly diagonally dominant  $M$ -matrix by rows, then for  $A^{-1} = (\alpha_{ij})$ , we have*

$$\alpha_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j, i} |a_{jk}| r_k}{a_{jj}} \alpha_{ii}, \quad \text{for all } j \neq i.$$

**Theorem 4.1** *If  $A = (a_{ij})$  and  $B = (b_{ij})$  are two nonsingular  $M$ -matrices and  $B^{-1} = (\beta_{ij})$ ,  $s_i = \max_{j \neq i} |a_{ij}|$ , then*

$$\begin{aligned} \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 \\ & + 4s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \Big\}. \end{aligned} \quad (4.5)$$

**Proof.** If  $A$  is an  $M$ -matrix, by Lemmas (4.1-4.2), there exists a positive diagonal matrix  $D$  such that  $D^{-1}AD$  is a strictly diagonally dominant  $M$ -matrix by rows.

**Case 1.** Suppose that  $A \circ B^{-1}$  is irreducible. Obviously  $A$  and  $B$  are also irreducible. Since  $A - \tau(A)I$  is an irreducible nonsingular  $M$ -matrix, then  $a_{ii} - \tau(A) > 0$ ,  $\forall i \in N$ , and there exists a positive vector  $u = (u_1, u_2, \dots, u_n)$  such that

$$Au = \tau(A)u,$$

where  $u = \text{diag}(u_1, u_2, \dots, u_n)$ ,  $u_i > 0$ , and then

$$a_{ii} + \sum_{j \neq i} \frac{a_{ji}u_j}{u_i} = \tau(A).$$

Define  $U = \text{diag}(u_1, u_2, \dots, u_n)$ ,  $C = U^{-1}AU$ , then we have that

$$C = (\tilde{a}_{ij}) = U^{-1}AU = \begin{pmatrix} a_{11} & \frac{a_{12}u_1}{u_2} & \dots & \frac{a_{1n}u_1}{u_n} \\ \frac{a_{21}u_2}{u_1} & a_{22} & \dots & \frac{a_{2n}u_2}{u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}u_n}{u_1} & \frac{a_{n2}u_n}{u_2} & \dots & a_{nn} \end{pmatrix}$$

is an irreducible nonsingular  $M$ -matrix.

By Lemma 2.2,

$$U^{-1}(A \circ B^{-1})U = (U^{-1}AU) \circ B^{-1} = C \circ B^{-1},$$

i.e.,  $\tau(A \circ B^{-1}) = \tau(C \circ B^{-1})$ .

By the inequality (2.4) and  $0 \leq \tau(A \star B) \leq a_{ii}b_{ii}$  (see [5]), for any  $j \neq i \in N$ ,

we have

$$\begin{aligned} |\tau(A \circ B^{-1}) - a_{ii}\beta_{ii}| |\tau(A \circ B^{-1}) - a_{jj}\beta_{jj}| &\leq \sum_{k \neq i} |\tilde{a}_{ki}| \beta_{ki} \sum_{l \neq j} |\tilde{a}_{lj}| \beta_{lj} \\ &\leq \sum_{k \neq i} |\tilde{a}_{ki}| \frac{b_{ki} + \sum_{u \neq k, i} |b_{ku}| r_u}{b_{kk}} \beta_{ii} \sum_{l \neq j} |\tilde{a}_{lj}| \frac{b_{lj} + \sum_{v \neq l, j} |b_{lv}| r_v}{b_{ll}} \beta_{jj} \\ &= \sum_{k \neq i} |\tilde{a}_{ki}| s_{ki} \beta_{ii} \sum_{l \neq j} |\tilde{a}_{lj}| s_{lj} \beta_{jj} \\ &\leq \sum_{k \neq i} |\tilde{a}_{ki}| s_i \beta_{ii} \sum_{l \neq j} |\tilde{a}_{lj}| s_j \beta_{jj} \\ &= \sum_{k \neq i} \frac{|a_{ki}| u_k}{u_j} s_i \beta_{ii} \sum_{l \neq j} \frac{|a_{jl}| u_j}{u_l} s_j \beta_{jj} \\ &= s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A)) (a_{jj} - \tau(A)). \end{aligned} \tag{4.6}$$

Thus, by solving the quadratic inequality (4.6), we obtain that

$$\begin{aligned} \tau(A \circ B^{-1}) &\geq \frac{1}{2} \left\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A))(a_{jj} - \tau(A))]^{\frac{1}{2}} \right\} \\ &\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A))(a_{jj} - \tau(A))]^{\frac{1}{2}} \right\}. \end{aligned}$$

i.e., the conclusion (4.5) holds.

**Case 2.** If  $A \circ B^{-1}$  is reducible, then one denotes by  $P = (p_{ij})$  the  $n \times n$  permutation matrix with

$$p_{12} = p_{23} = \dots = p_{n-1,n} = p_{n,1} = 1,$$

the remaining  $p_{ij}$  zero, then both  $A - \varepsilon P$  and  $B - \varepsilon P$  are irreducible nonsingular  $M$ -matrices for any sufficiently small positive real number  $\varepsilon$ . Now we substitute  $A - \varepsilon P$  and  $B - \varepsilon P$  for  $A$  and  $B$ , respectively from the previous Case, and then letting  $\varepsilon \rightarrow 0$ , the result (2.6) follows by continuity.  $\square$

**Example 4.1** ([11]). Let  $A$  and  $B$  be nonsingular  $M$ -matrices:

$$A = (a_{ij}) = \begin{pmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{pmatrix}, \quad B = (b_{ij}) = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}.$$

By direct calculation,  $\tau(A \circ B^{-1}) = 0.2148$ .

According to (4.1), we have

$$\tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii} = 0.07.$$

If we apply (4.2) and (4.3), we get

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A \rho(J_B))}{1 + \rho^2(J_B)} \min_i \frac{b_{ii}}{a_{ii}} = 0.0707,$$

and

$$\tau(A \circ B^{-1}) \geq \min_i \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} = 0.08.$$

According to (4.4)

$$\begin{aligned} \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \{ & a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - [(a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 \\ & + 4a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B)]^{\frac{1}{2}} \} = 0.1524. \end{aligned}$$

If we apply Theorem 4.1, we obtain that

$$\begin{aligned} \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \{ & a_{ii} \beta_{ii} + a_{jj} \beta_{jj} + [(a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 \\ & + 4s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A))(a_{jj} - \tau(A))]^{\frac{1}{2}} \} = 0.1929. \end{aligned}$$

The example shows that the bound in Theorem 4.1 is better than the existing bounds.

## 5 Inequalities for the Fan product of several $M$ -matrices

Firstly, let us recall the following lemmas.

**Lemma 5.1** ([7]). *Let  $A$  be an irreducible nonsingular  $M$ -matrix, if  $AZ \geq kZ$  for a nonnegative nonzero vector  $Z$ , then  $k \leq \tau(A)$ .*

**Lemma 5.2** ([6]). Let  $x_j = (x_j(1), \dots, x_j(n))^T \geq 0$ ,  $j \in \{1, 2, \dots, m\}$ , if  $P_j > 0$  and  $\sum_{k=1}^m \frac{1}{P_k} \geq 1$ , then we have

$$\sum_{i=1}^n \prod_{j=1}^m x_j(i) \leq \prod_{j=1}^m \left\{ \sum_{i=1}^n [x_j(i)]^{P_j} \right\}^{\frac{1}{P_j}}. \quad (5.1)$$

Next, according to these results, we expand the inequality (3.2) of the Fan product of two matrices to the Fan product of several matrices. One can obtain the following result:

**Theorem 5.1** For any matrices  $A_k \in M_n$ , and positive integers  $P_k$  with  $\sum_{k=1}^m \frac{1}{P_k} \geq 1$ ,  $k \in \{1, 2, \dots, m\}$ , we have that

$$\tau(A_1 \star A_2 \cdots \star A_m) \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m A_k(i, i) - \prod_{k=1}^m [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}} \right\}. \quad (5.2)$$

**Proof.** It is quite evident that the (5.2) holds with the equality for  $n = 1$ . Below we assume that  $n \geq 2$ .

**Case 1.** Let  $A_1 \star A_2 \cdots \star A_m$  be an irreducible nonsingular  $M$ -matrix, thus  $A_k$  is irreducible,  $k \in \{1, 2, \dots, m\}$ , we can obtain that  $A_k^{(P_k)}$  is also irreducible. Let  $u_k^{(P_k)} = (u_k(1)^{P_k}, \dots, u_k(n)^{P_k})^T > 0$  be a right Perron eigenvector of  $A_k^{(P_k)}$ , and  $u_k = (u_k(1), \dots, u_k(n))^T > 0$ , thus for any  $i \in N$ , we have that

$$A_k^{(P_k)} u_k^{(P_k)} = \tau(A_k^{(P_k)}) u_k^{(P_k)},$$

$$A_k(i, i)^{P_k} u_k(i)^{P_k} - \sum_{j \neq i} |A_k(i, j)^{P_k}| u_k(j)^{P_k} = \tau(A_k^{(P_k)}) u_k(i)^{P_k},$$

and

$$\sum_{j \neq i} |A_k(i, j)^{P_k}| u_k(j)^{P_k} = \left( A_k(i, i)^{P_k} - \tau(A_k^{(P_k)}) \right) u_k(i)^{P_k}. \quad (5.3)$$

Denote  $C = A_1 \star A_2 \cdots \star A_m$ ,  $Z = u_1 \star u_2 \cdots \star u_m = (Z(1), \dots, Z(n))^T > 0$ , thus  $Z(i) = \prod_{k=1}^m u_k(i)$ . By the Lemma 5.2 and (5.3), we get that

$$\begin{aligned} (CZ)_i &= \left( \prod_{k=1}^m A_k(i, i) \right) Z(i) - \left( \sum_{j \neq i} \prod_{k=1}^m |A_k(i, j)| \right) Z(j) \\ &= \left( \prod_{k=1}^m A_k(i, i) \right) Z(i) - \sum_{j \neq i} \prod_{k=1}^m \left( |A_k(i, j)| u_k(j) \right) \\ &\geq \left( \prod_{k=1}^m A_k(i, i) \right) Z(i) - \prod_{k=1}^m \left\{ \sum_{j \neq i} [|A_k(i, j)| u_k(j)]^{P_k} \right\}^{\frac{1}{P_k}} \quad (\text{by the equality (5.3)}) \\ &= \left( \prod_{k=1}^m A_k(i, i) \right) Z(i) - \prod_{k=1}^m \left\{ [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})] u_k(i)^{P_k} \right\}^{\frac{1}{P_k}} \\ &= \left\{ \prod_{k=1}^m A_k(i, i) - \prod_{k=1}^m [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})] \right\}^{\frac{1}{P_k}} Z(i). \end{aligned}$$

According to the Lemma 5.1, we obtain that

$$\tau(A_1 \star A_2 \cdots \star A_m) \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^m A_k(i, i) - \prod_{k=1}^m [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}} \right\}.$$

**Case 2.** If  $A_1 \star A_2 \cdots \star A_m$  is reducible, where  $A_i$  ( $i = 1, 2, \dots, m$ ) are nonsingular  $M$ -matrices. Similarly, let  $P = (p_{ij})$  be the  $n \times n$  permutation matrix with  $p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1$ , the remaining  $p_{ij}$  zero, then  $A_k - \varepsilon P$  is an irreducible nonsingular  $M$ -matrix for any chosen positive real number  $\varepsilon$ . Note that  $A_k - \varepsilon P$  is a continuous function on  $\varepsilon$ . Now we substitute  $A_k - \varepsilon P$  for  $A_k$ , in the previous Case 1, and then letting  $\varepsilon \rightarrow 0$ , the result (5.2) follows by continuity.  $\square$

**Remark 4.1.** If we take  $m = 2$  in Theorem 4.1, one can obtain the following results:

- If  $p_1 = p_2 = 1$ ,  $A_1 = A = (a_{ij})$ ,  $A_2 = B = (b_{ij})$ , we have that

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - (a_{ii} - \tau(A))(b_{ii} - \tau(B)) \right\},$$

which is just the inequality (3.2).

- If  $p_1 = p_2 = 2$ ,  $A_1 = A = (a_{ij})$ ,  $A_2 = B = (b_{ij})$ , then

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - [a_{ii}^2 - \tau(A \star A)]^{\frac{1}{2}} [b_{ii}^2 - \tau(B \star B)]^{\frac{1}{2}} \right\}. \quad (5.4)$$

In addition, by using the inequalities of arithmetic and geometric means, we may obtain that

$$a_{ii}^2 \tau(B \star B) + b_{ii}^2 \tau(A \star A) \geq 2a_{ii}b_{ii} [\tau(A \star A) \tau(B \star B)]^{\frac{1}{2}},$$

so

$$(a_{ii}^2 - \tau(A \star A))(b_{ii}^2 - \tau(B \star B)) \leq \left\{ a_{ii}b_{ii} - [\tau(A \star A) \tau(B \star B)]^{\frac{1}{2}} \right\}^2. \quad (5.5)$$

Since for any  $A, B \in M_n$ ,  $\tau(A \star B) \geq \tau(A)\tau(B)$  (see [1] or (3.1)), then, by (5.5), we have that

$$a_{ii}b_{ii} - \left[ (a_{ii}^2 - \tau(A \star A))(b_{ii}^2 - \tau(B \star B)) \right]^{\frac{1}{2}} \geq [\tau(A \star A) \tau(B \star B)]^{\frac{1}{2}} \geq \tau(A)\tau(B).$$

That is, the bound in (5.2) is better than the bound in (3.1).

- If  $p_1 = 1, p_2 = 2$ ,  $A_1 = A = (a_{ij})$ ,  $A_2 = B = (b_{ij})$ , then we get

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - [a_{ii} - \tau(A)][b_{ii}^2 - \tau(B \star B)]^{\frac{1}{2}} \right\}.$$

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